CSC D70: Compiler Optimization
Register Allocation

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The content of this lecture is adapted from the lectures of Todd Mowry and Phillip Gibbons
Register Allocation and Coalescing

- Introduction
- Abstraction and the Problem
- Algorithm
- Spilling
- Coalescing

Reading: ALSU 8.8.4
Motivation

• **Problem**
  – Allocation of variables (pseudo-registers) to hardware registers in a procedure

• **A very important optimization!**
  – Directly reduces running time
    • (memory access $\rightarrow$ register access)
  – Useful for other optimizations
    • e.g. CSE assumes old values are kept in registers.
Goals

• Find an allocation for all pseudo-registers, if possible.

• If there are not enough registers in the machine, choose registers to spill to memory
Register Assignment Example

A = ...
IF A goto L1

B = ...
  = A
D =
  = B + D

L1: C = ...
  = A
D =
  = C + D

• Find an assignment (no spilling) with only 2 registers
  – A and D in one register, B and C in another one
• What assumptions?
  – After assignment, no use of A & (and only one of B and C used)
An Abstraction for Allocation & Assignment

• **Intuitively**
  – Two pseudo-registers *interfere* if at some point in the program they cannot both occupy the same register.

• **Interference graph**: an undirected graph, where
  – nodes = pseudo-registers
  – there is an edge between two nodes if their corresponding pseudo-registers interfere

• **What is not represented**
  – Extent of the interference between uses of different variables
  – Where in the program is the interference

Interfere many times vs. once

E.g., cold path vs. hot path
Register Allocation and Coloring

• A graph is \textbf{n-colorable} if:
  – every node in the graph can be colored with one of the \textit{n} colors such that two adjacent nodes do not have the same color.

• Assigning \textit{n} register (without spilling) = Coloring with \textit{n} colors
  – assign a node to a register (color) such that no two adjacent nodes are assigned the same registers (colors)

• Is spilling necessary? = Is the graph \textbf{n-colorable}?

• To determine if a graph is \textbf{n-colorable} is \textbf{NP-complete}, for \textit{n}>2
  – Too expensive
  – Heuristics
Algorithm

**Step 1. Build an interference graph**
  a. refining notion of a node
  b. finding the edges

**Step 2. Coloring**
  – use heuristics to try to find an n-coloring
    • **Success:**
      – colorable and we have an assignment
    • **Failure:**
      – graph not colorable, or
      – graph is colorable, but it is too expensive to color
Step 1a. Nodes in an Interference Graph

A = ...
IF A goto L1

B = ...
  = A
D = 
  = B + D

L1: C = ...
  = A
D = 
  = D + C

A = 2

Should we add A-D edge?
No, since new def of A
Live Ranges and Merged Live Ranges

• Motivation: to create an interference graph that is easier to color
  – Eliminate interference in a variable’s “dead” zones.
  – Increase flexibility in allocation:
    • can allocate same variable to different registers
• A live range consists of a definition and all the points in a program in which that definition is live.
  – How to compute a live range?
• Two overlapping live ranges for the same variable must be merged

Diagram:
```
  a = ...
     /  |
    /   |
   /     |
  a = ...
     /  |
    /   |
   /     |
  ... = a
```
Example (Revisited)

Live Variables
Reaching Definitions

\[
\begin{align*}
A &= \ldots \ (A_1) \\
\text{IF } A \text{ goto L1} & \quad \{A\} \quad \{A_1\} \\
& \quad \{A\} \quad \{A_1\} \\
B &= \ldots \ (B_1) \\
& = A \\
D &= B \ (D_2) \\
\end{align*}
\]

\[
\begin{align*}
L1: \\
C &= \ldots \ (C_1) \\
& = A \\
D &= \ldots \ (D_1) \\
\end{align*}
\]

\[
\begin{align*}
A &= 2 \ (A_2) \\
\end{align*}
\]

\[
\begin{align*}
\{A\} \quad \{A_1\} \\
\{A,C\} \quad \{A_1,C_1\} \\
\{C\} \quad \{A_1,C_1\} \\
\{D\} \quad \{A_1,C_1,D_1\} \\
\end{align*}
\]

\[
\begin{align*}
\{A,D\} \quad \{A_2,B_1,C_1,D_1,D_2\} \\
\{A_1,B_1,C_1,D_1,D_2\} \\
\end{align*}
\]

\[
\begin{align*}
\{A,D\} \quad \{A_2,B_1,C_1,D_1,D_2\} \\
\{D\} \quad \{A_2,B_1,C_1,D_1,D_2\} \\
\end{align*}
\]

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\{A,D\} \quad \{A_2,B_1,C_1,D_1,D_2\} \\
\{D\} \quad \{A_2,B_1,C_1,D_1,D_2\} \\
\end{align*}
\]

Merge
Merging Live Ranges

• **Merging definitions into equivalence classes**
  – Start by putting each definition in a different equivalence class
  – Then, **for each point** in a program:
    • if (i) variable is live, and (ii) there are multiple reaching definitions for the variable, then:
      – merge the equivalence classes of all such definitions into one equivalence class
    • *(Sound familiar?)*

• **From now on, refer to merged live ranges simply as live ranges**
  – merged live ranges are also known as “webs”
SSA Revisited: What Happens to $\Phi$ Functions

• Now we see why it is unnecessary to “implement” a $\Phi$ function
  – $\Phi$ functions and SSA variable renaming simply turn into merged live ranges

• When you encounter: $x_4 = \Phi(x_1, x_2, x_3)$
  – merge $x_1, x_2, x_3$, and $x_4$ into the same live range
  – delete the $\Phi$ function

• Now you have effectively converted back out of SSA form
Step 1b. Edges of Interference Graph

- **Intuitively:**
  - Two live ranges (necessarily of different variables) may interfere if they overlap at some point in the program.
  - Algorithm:
    - At each point in the program:
      - enter an edge for every pair of live ranges at that point.

- **An optimized definition & algorithm for edges:**
  - Algorithm:
    - check for interference only at the start of each live range
    - Faster
    - Better quality
Live Range Example 2

Because ranges overlap: Won’t assign A and B to same register (even though would have been ok: path sensitive vs. path insensitive analysis)
Step 2. Coloring

• Reminder: coloring for $n > 2$ is NP-complete

• Observations:
  – a node with $\text{degree} < n \Rightarrow$
    • can always color it successfully, given its neighbors’ colors
  
  – a node with $\text{degree} = n \Rightarrow$
    • can only color if at least two neighbors share same color
  
  – a node with $\text{degree} > n \Rightarrow$
    • maybe, not always
Coloring Algorithm

**Algorithm:**
- Iterate until stuck or done
  - Pick any node with degree < n
  - Remove the node and its edges from the graph
- If done (no nodes left)
  - reverse process and add colors

**Example (n = 3):**

- **Note:** degree of a node may drop in iteration
- Avoids making arbitrary decisions that make coloring fail
More details

- **Apply coloring heuristic**
  
  Build interference graph
  Iterate until there are no nodes left
  
  - If there exists a node v with less than n neighbor
    - push v on register allocation stack
  
  else
  
  - return *(coloring heuristics fail)*
  
  remove v and its edges from graph

- **Assign registers**

  While stack is not empty
  
  - Pop v from stack
  
  Reinsert v and its edges into the graph
  
  Assign v a color that differs from all its neighbors
What Does Coloring Accomplish?

• **Done:**
  – colorable, also obtained an assignment

• **Stuck:**
  – colorable or not?

![Diagram of a graph with vertices labeled E, A, B, C, D, and E.](image)
Extending Coloring: Design Principles

• **A pseudo-register is**
  – Colored successfully: allocated a hardware register
  – Not colored: left in memory

• **Objective function**
  – Cost of an uncolored node:
    • proportional to number of uses/definitions (dynamically)
    • estimate by its loop nesting
  – Objective: **minimize sum of cost of uncolored nodes**

• **Heuristics**
  – **Benefit of spilling** a pseudo-register:
    • increases colorability of pseudo-registers it interferes with
    • can **approximate by its degree in interference graph**
  – **Greedy heuristic**
    • spill the pseudo-register with lowest cost-to-benefit ratio, whenever spilling is necessary
Spilling to Memory

- **CISC architectures**
  - can operate on data in memory directly
  - memory operations are slower than register operations

- **RISC architectures**
  - machine instructions can only apply to registers
  - **Use**
    - must first load data from memory to a register before use
  - **Definition**
    - must first compute RHS in a register
    - store to memory afterwards
  - Even if spilled to memory, needs a register at time of use/definition
Chaitin: Coloring and Spilling

• **Identify spilling**
  
  Build interference graph
  Iterate until there are no nodes left
  
  *If there exists a node v with less than n neighbor*
  
  place v on stack to register allocate
  else
  
  *v = node with highest degree-to-cost ratio*
  mark v as spilled
  remove v and its edges from graph

• **Spilling may require use of registers; change interference graph**
  
  While there is spilling
  rebuild interference graph and perform step above

• **Assign registers**
  
  While stack is not empty
  Remove v from stack
  Reinsert v and its edges into the graph
  Assign v a color that differs from all its neighbors
Spilling

• What should we spill?
  – Something that will eliminate a lot of interference edges
  – Something that is used infrequently
  – Maybe something that is live across a lot of calls?

• One Heuristic:
  – spill cheapest live range (aka “web”)
  – Cost = \((# \text{ defs} & \text{ uses}) \times 10^{\text{loop-nest-depth}}\)/degree
Quality of Chaitin’s Algorithm

• Giving up too quickly

• N=2

• An optimization: “Prioritize the coloring”
  – Still eliminate a node and its edges from graph
  – Do not commit to “spilling” just yet
  – Try to color again in assignment phase.
Splitting Live Ranges

- **Recall:** Split pseudo-registers into live ranges to create an interference graph that is easier to color
  - Eliminate interference in a variable’s “dead” zones.
  - Increase flexibility in allocation:
    - can allocate same variable to different registers

```
A = ...
IF A goto L1
B = ...
= A
D = B
A = D

L1: C = ...
D = A
= C
= A

```

```
A1
B
D
C
A2
```
Insight

• Split a live range into smaller regions (by paying a small cost) to create an interference graph that is easier to color
  – Eliminate interference in a variable’s “nearly dead” zones.
    • Cost: Memory loads and stores
      – Load and store at boundaries of regions with no activity
    • # active live ranges at a program point can be > # registers

– Can allocate same variable to different registers
  • Cost: Register operations
    – a register copy between regions of different assignments
  • # active live ranges cannot be > # registers
Examples

Example 1:

FOR i = 0 TO 10
FOR j = 0 TO 10000
    A = A + ...  
    (does not use B)
FOR j = 0 TO 10000
    B = B + ...
    (does not use A)

Example 2:

\[
\begin{align*}
    a &= b + c \\
    b &= a + b \\
    c &= a + c \\
    b &= b + c
\end{align*}
\]
Example 1

```
FOR i = 0 TO 10
  FOR j = 0 TO 10000
    A = A + ...
    (does not use B)
  FOR j = 0 TO 10000
    B = B + ...
    (does not use A)
```
Example 2

\[
\begin{align*}
\text{Example 2} & \\
\text{Diagram A} & \\
b &= a + b \\
c &= \text{else} \\
\text{Diagram B} & \\
a &= a \\
b &= a + b \\
c &= a + c \\
\text{Diagram C} & \\
a &= a \\
b &= a + b \\
c &= a + c \\
\end{align*}
\]
Live Range Splitting

• When do we apply live range splitting?
• Which live range to split?
• Where should the live range be split?
• How to apply live-range splitting with coloring?

  – Advantage of coloring:
    • defers arbitrary assignment decisions until later
  – When coloring fails to proceed, may not need to split live range
    • degree of a node >= n does not mean that the graph definitely is not colorable
  – Interference graph does not capture positions of a live range
One Algorithm

• **Observation**: spilling is absolutely necessary if
  – number of live ranges active at a program point > n

• **Apply live-range splitting before coloring**
  – Identify a point where number of live ranges > n
  – For each live range active around that point:
    • find the outermost “block construct” that does not access the variable
  – Choose a live range with the largest inactive region
  – Split the inactive region from the live range
Summary

• **Problems:**
  – Given n registers in a machine, is spilling avoided?
  – Find an assignment for all pseudo-registers, whenever possible.

• **Solution:**
  – Abstraction: an *interference graph*
    • nodes: live ranges
    • edges: presence of live range at time of definition
  – Register Allocation and Assignment problems
    • equivalent to *n-colorability* of interference graph
      \(\rightarrow\) NP-complete
  – Heuristics to find an assignment for n colors
    • successful: colorable, and finds assignment
    • not successful: colorability unknown & no assignment
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Let’s Focus on Copy Instructions

- Optimizations that help optimize away copy instructions:
  - Copy Propagation
  - Dead Code Elimination

- Can all copy instructions be eliminated using this pair of optimizations?

```
X = A + B;
...  
Y = X;
...
Z = Y + 4;

X = A + B;
...
// deleted
...
Z = X + 4;
```
Example Where Copy Propagation Fails

- Use of copy target has multiple (conflicting) reaching definitions

```plaintext
X = A + B;
Y = C;
Y = X;
Z = Y + 4;
```
Another Example Where the Copy Instruction Remains

- Copy target ($Y$) still live even after some successful copy propagations

- **Bottom line:**
  - copy instructions may still exist when we perform register allocation

```
X = A + B;
Y = X;
Z = Y + 4;

Y = ...;

C = Y + D;
```

Can substitute $X$ for $Y$ here

But not here
Copy Instructions and Register Allocation

• What clever thing might the register allocator do for copy instructions?

• If we can assign both the source and target of the copy to the same register:
  – then we don’t need to perform the copy instruction at all!
  – the copy instruction can be removed from the code
    • even though the optimizer was unable to do this earlier

• One way to do this:
  – treat the copy source and target as the same node in the interference graph
    • then the coloring algorithm will naturally assign them to the same register
  – this is called “coalescing”
Simple Example: Without Coalescing

- Without coalescing, $X$ and $Y$ can end up in different registers
  - cannot eliminate the copy instruction

```c
X = ...;
A = 5;
Y = X;
B = A + 2;
Z = Y + B;
return Z;
```
Example Revisited: With Coalescing

With coalescing, \( X \) and \( Y \) are now guaranteed to end up in the same register— the copy instruction can now be eliminated.

Great! So should we go ahead and do this for every copy instruction?

```
X = ...;
A = 5;
Y = X;  // Note: This line is crossed out, indicating it is no longer necessary.
B = A + 2;
Z = Y + B;
return Z;
```
Should We Coalesce $X$ and $Y$ In This Case?

- It is legal to coalesce $X$ and $Y$ for a “$Y = X$” copy instruction iff:
  - initial definition of $Y$’s live range is this copy instruction, AND
  - the live ranges of $X$ and $Y$ do not interfere otherwise

- But just because it is legal doesn’t mean that it is a good idea…

No! That would result in incorrect behavior if this branch is taken.

```
X = A + B;
Y = X;

Z = Y + X;
X = 2;
```

```plaintext
X = A + B;
Y = X;

Z = Y + X;
X = 2;
```
Why Coalescing May Be Undesirable

\[ X = A + B; \]
\[ \ldots \text{// 100 instructions} \]
\[ Y = X; \]
\[ \ldots \text{// 100 instructions} \]
\[ Z = Y + 4; \]

- What is the likely impact of coalescing \( X \) and \( Y \) on:
  - live range size(s)?
    - recall our discussion of live range splitting
  - colorability of the interference graph?
- Fundamentally, coalescing adds further constraints to the coloring problem
  - doesn’t make coloring easier; may make it more difficult
- If we coalesce in this case, we may:
  - save a copy instruction, BUT
  - cause significant spilling overhead if we can no longer color the graph
When to Coalesce

• Goal when coalescing is legal:
  – coalesce *unless* it would make a colorable graph *non-colorable*

• The bad news:
  – predicting colorability is tricky!
    • it depends on the shape of the graph
    • graph coloring is NP-hard

• **Example**: assuming 2 registers, should we coalesce *X* and *Y*?

![Diagram of 2-colorable graph](image)

![Diagram of Not 2-colorable graph](image)
Representing Coalescing Candidates in the Interference Graph

• To decide whether to coalesce, we augment the interference graph
• Coalescing candidates are represented by a new type of interference graph edge:
  – dotted lines: coalescing candidates
    • *try* to assign vertices the same color
    – (unless that is problematic, in which case they can be given different colors)
  – solid lines: interference
    • vertices *must* be assigned different colors

```plaintext
X = ...;
A = 5;
Y = X;
B = A + 2;
Z = Y + B;
return Z;
```
How Do We Know When Coalescing Will Not Cause Spilling?

- **Key insight:**
  - Recall from the coloring algorithm:
    - we can always successfully N-color a node if its degree is < N

- To ensure that coalescing does not cause spilling:
  - check that the degree < N invariant is still locally preserved after coalescing
    - if so, then coalescing won’t cause the graph to become non-colorable
  - no need to inspect the entire interference graph, or do trial-and-error

- **Note:**
  - We do NOT need to determine whether the full graph is colorable or not
  - Just need to check that coalescing does not cause a colorable graph to become non-colorable
Simple and Safe Coalescing Algorithm

• We can safely coalesce nodes X and Y if $(|X| + |Y|) < N$
  – Note: $|X| = \text{degree of node } X \text{ counting interference (not coalescing) edges}$

• Example:

  $X$ • • $Y$

  $(|X| + |Y|) = (1 + 2) = 3$

  $X/Y$

  Degree of coalesced node can be no larger than 3

  – if $N \geq 4$, it would always be safe to coalesce these two nodes
    • this cannot cause new spilling that would not have occurred with the original graph
  – if $N < 4$, it is unclear

  *How can we (safely) be more aggressive than this?*
What About This Example?

• Assume $N = 3$
• Is it safe to coalesce $X$ and $Y$?

![Diagram of nodes X, Y, A, B, Z with connections showing degree of coalesced node]

$(|X| + |Y|) = (1 + 2) = 3$
(Not less than $N$)

• **Notice:** $X$ and $Y$ share a common (interference) neighbor: node $A$
  – hence the degree of the coalesced $X/Y$ node is actually 2 (not 3)
  – therefore coalescing $X$ and $Y$ is guaranteed to be safe when $N = 3$
• How can we adjust the algorithm to capture this?
Another Helpful Insight

• Colors are not assigned until nodes are popped off the stack
  – nodes with degree < N are pushed on the stack first
  – when a node is popped off the stack, we know that it can be colored
    • because the number of potentially conflicting neighbors must be < N

• Spilling only occurs if there is no node with degree < N to push on the stack

• Example: (N=2)
Another Helpful Insight

|\textbf{X}| = 5
|\textbf{Y}| = 5

2-colorable after coalescing $\textbf{X}$ and $\textbf{Y}$?
Building on This Insight

- When would coalescing cause the stack pushing (aka “simplification”) to get stuck?
  1. coalesced node must have a degree $\geq N$
     - otherwise, it can be pushed on the stack, and we are not stuck
  2. AND it must have at least $N$ neighbors that each have a degree $\geq N$
     - otherwise, all neighbors with degree $< N$ can be pushed before this node
       - reducing this node’s degree below $N$ (and therefore we aren’t stuck)

- To coalesce more aggressively (and safely), let’s exploit this second requirement
  - which involves looking at the degree of a coalescing candidate’s neighbors
    - not just the degree of the coalescing candidates themselves
Briggs’s Algorithm

• Nodes X and Y can be coalesced if:
  – \((\text{number of neighbors of } X/Y \text{ with degree } \geq N) < N\)

• Works because:
  – all other neighbors can be pushed on the stack before this node,
  – and then its degree is < N, so then it can be pushed
  – **Example**: \((N = 2)\)
Briggs’s Algorithm

- Nodes $X$ and $Y$ can be coalesced if:
  - (number of neighbors of $X/Y$ with degree $\geq N) < N$
- More extreme example: ($N = 2$)

<table>
<thead>
<tr>
<th>X/Y</th>
<th>J</th>
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<tbody>
<tr>
<td>I</td>
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<td>A</td>
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</tbody>
</table>
George’s Algorithm

Motivation:
• imagine that $X$ has a very high degree, but $Y$ has a much smaller degree
  – (perhaps because $X$ has a large live range)

• With Briggs’s algorithm, we would inspect all neighbors both $X$ and $Y$
  – but $X$ has a lot of neighbors!
• Can we get away with just inspecting the neighbors of $Y$?
  – showing that coalescing makes coloring no worse than it was given $X$?
George’s Algorithm

• Coalescing $X$ and $Y$ does no harm if:
  - foreach neighbor $T$ of $Y$, either:
    1. degree of $T$ is $<N$, or $\leftarrow$ similar to Briggs: $T$ will be pushed before $X/Y$
    2. $T$ interferes with $X$ $\leftarrow$ hence no change compared with coloring $X$

• Example: $(N=2)$
Summary

• **Coalescing** can enable register allocation to **eliminate copy instructions**
  – if both source and target of copy can be allocated to the same register
• However, coalescing must be applied with care to **avoid causing register spilling**
• Augment the interference graph:
  – dotted lines for coalescing candidate edges
  – try to allocate to same register, unless this may cause spilling
• **Coalescing Algorithms**:
  – simply based upon **degree of coalescing candidate nodes** (**X** and **Y**)
  – Briggs’s algorithm
    • look at degree of neighboring nodes of **X** and **Y**
  – George’s algorithm
    • asymmetrical: **look at neighbors of Y** (degree and interference with **X**)